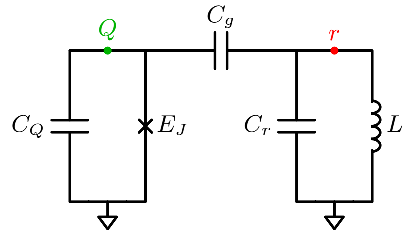

The capacitance matrix is given by

$$\mathbf{C} = \begin{pmatrix} C_Q + C_g & -C_g \\ -C_g & C_r + C_g \end{pmatrix}.$$

The inverse is

$$\begin{aligned} \mathbf{C}^{-1} &= \frac{1}{\left(C_Q + C_g\right)\left(C_r + C_g\right) - C_g^2} \begin{pmatrix} C_r + C_g & C_g \\ C_g & C_Q + C_g \end{pmatrix} \\ &= \frac{1}{C_Q C_r + C_Q C_g + C_r C_g} \begin{pmatrix} C_r + C_g & C_g \\ C_g & C_Q + C_g \end{pmatrix}. \end{aligned}$$

The Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\mathbf{q}^T \mathbf{C}^{-1} \mathbf{q} + U(\Phi) \\ &= \frac{1}{2}\frac{1}{C_Q C_r + C_Q C_g + C_r C_g} \begin{pmatrix} q_Q & q_r \end{pmatrix} \begin{pmatrix} C_r + C_g & C_g \\ C_g & C_Q + C_g \end{pmatrix} \begin{pmatrix} q_Q \\ q_r \end{pmatrix} + \frac{\Phi_r}{2L} - E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi_Q\right) \\ &= \frac{1}{2}\frac{1}{C_Q C_r + C_Q C_g + C_r C_g} \left[\left(C_r + C_g\right) q_Q^2 + \left(C_Q + C_g\right) q_r^2 + 2C_g q_Q q_r\right] + \frac{\Phi_r}{2L} - E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi_Q\right). \end{aligned}$$

Separating this into the three Hamiltonian terms, we get

$$\begin{aligned} H_{\text{transmon}} &= \frac{1}{2}\frac{C_r + C_g}{C_Q C_r + C_Q C_g + C_r C_g} q_Q^2 - E_J \cos\left(\frac{2\pi}{\Phi_0}\Phi_Q\right) \\ H_{\text{resonator}} &= \frac{1}{2}\frac{C_Q + C_g}{C_Q C_r + C_Q C_g + C_r C_g} q_r^2 + \frac{\Phi_r^2}{2L} \\ H_{\text{int}} &= \frac{C_g}{C_Q C_r + C_Q C_g + C_r C_g} q_Q q_r. \end{aligned}$$

We can now define the dimensionless charge number and flux operators

$$n = \frac{q}{2e}, \qquad \varphi = \frac{2\pi}{\Phi_0}\Phi = \frac{\Phi}{\varphi_0},$$

to get

$$\begin{aligned} H_{\text{transmon}} &= \frac{1}{2}\frac{C_r + C_g}{\det \mathbf{C}} 4e^2 n_Q^2 - E_J \cos\left(\varphi_Q\right) \\ &= \frac{4e^2}{2\widetilde{C}_Q} n_Q^2 - E_J \cos\left(\varphi_Q\right) \\ H_{\text{resonator}} &= \frac{1}{2}\frac{C_Q + C_g}{\det \mathbf{C}} 4e^2 n_r^2 + \frac{\varphi_0^2}{2L}\varphi_r^2 \\ &= \frac{4e^2}{2\widetilde{C}_r} n_r^2 + \frac{\varphi^2}{2L}\varphi_r^2 \\ H_{\text{int}} &= \frac{C_g}{\det \mathbf{C}} 4e^2 n_Q n_r, \end{aligned}$$

where we have used the fact that the denominator is the determinant of the capacitance matrix

$$\det \mathbf{C} = \left(C_Q + C_g\right)\left(C_r + C_g\right) - C_g^2 = C_Q C_r + C_Q C_g + C_r C_g$$

and we have defined the capacitances

$$\widetilde{C}_Q = \frac{\det \mathbf{C}}{C_r + C_g}, \qquad \widetilde{C}_r = \frac{\det \mathbf{C}}{C_Q + C_g}.$$

We now define

$$E_C = \frac{e^2}{2\widetilde{C}}, \qquad E_L = \frac{\varphi_0^2}{L}$$

to get

$$H_{\text{transmon}} = 4E_{C,Q} n_Q^2 - E_J \cos(\varphi_Q), \qquad H_{\text{resonator}} = 4E_{C,r} n_r^2 + \frac{E_L}{2}\varphi_r^2.$$

We now quantize the Hamiltonian by promoting n and φ to operators and then writing them in terms of mode operators as

$$\hat{\varphi}_Q = \left(\frac{2E_{C,Q}}{E_J}\right)^{1/4} \left(\hat{b}^\dagger + \hat{b}\right), \qquad \hat{n}_Q = \frac{i}{2}\left(\frac{E_J}{2E_{C,Q}}\right)^{1/4} \left(\hat{b}^\dagger - \hat{b}\right)$$

for the transmon and

$$\hat{\varphi}_r = \left(\frac{2E_{C,r}}{E_L}\right)^{1/4} \left(\hat{a}^\dagger + \hat{a}\right), \qquad \hat{n}_r = \frac{i}{2}\left(\frac{E_L}{2E_{C,r}}\right)^{1/4} \left(\hat{a}^\dagger - \hat{a}\right).$$

We see that these satisfy the commutation relations as needed

$$\begin{aligned}
\left[\hat{\varphi}_Q, \hat{n}_Q\right] &= \frac{i}{2}\left(\frac{2E_{C,Q}}{E_J}\right)^{1/4}\left(\frac{E_J}{2E_{C,Q}}\right)^{1/4}\left[\hat{b}^\dagger + \hat{b}, \hat{b}^\dagger - \hat{b}\right] \\
&= \frac{i}{2}\cdot 2 \\
&= i, \\
\left[\hat{\varphi}_r, \hat{n}_r\right] &= \frac{i}{2}\left(\frac{2E_{C,r}}{E_L}\right)^{1/4}\left(\frac{E_L}{2E_{C,r}}\right)^{1/4}\left[\hat{a}^\dagger + \hat{a}, \hat{a}^\dagger - \hat{a}\right] \\
&= \frac{i}{2}\cdot 2 \\
&= i.
\end{aligned}$$

The resonator Hamiltonian can be written as

$$\begin{aligned}
\hat{H}_{\text{resonator}} &= 4E_{C,r}\frac{i^2}{4}\left(\frac{E_L}{2E_{C,r}}\right)^{1/2}\left(\hat{a}^\dagger - \hat{a}\right)^2 + \frac{E_L}{2}\left(\frac{2E_{C,r}}{E_L}\right)^{1/2}\left(\hat{a}^\dagger + \hat{a}\right)^2 \\
&= -\left(\frac{E_L E_{C,r}}{2}\right)^{1/2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\right) \\
&\quad + \left(\frac{E_L E_{C,r}}{2}\right)^{1/2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \\
&= \left(\frac{E_L E_{C,r}}{2}\right)^{1/2}\cdot 2\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \\
&= \sqrt{2E_L E_{C,r}}\cdot 2\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&= \sqrt{8E_L E_{C,r}}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&= \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \qquad \left\{\omega_r = \frac{1}{\hbar}\sqrt{8E_L E_{C,r}}\right\}
\end{aligned}$$

The interaction Hamiltonian then becomes

$$\begin{aligned}
\hat{H}_{\text{int}} &= \frac{4e^2 C_g}{\det \mathbf{C}}\cdot\frac{i}{2}\left(\frac{E_J}{2E_{C,Q}}\right)^{1/4}\left(\hat{b}^\dagger - \hat{b}\right)\cdot\frac{i}{2}\left(\frac{E_L}{2E_{C,r}}\right)^{1/4}\left(\hat{a}^\dagger - \hat{a}\right) \\
&= -\frac{e^2 C_g}{\det \mathbf{C}}\left(\frac{E_J E_L}{4E_{C,Q}E_{C,r}}\right)^{1/4}\left(\hat{b}^\dagger - \hat{b}\right)\left(\hat{a}^\dagger - \hat{a}\right).
\end{aligned}$$

Writing this as $\hat{H}_{\text{int}} = g\left(\hat{a}^\dagger - \hat{a}\right)\left(\hat{b}^\dagger - \hat{b}\right)$, where the order of $\hat{a}$ and $\hat{b}$ have been switched since they commute with each other, we get

$$\begin{aligned}
g &= -\frac{e^2 C_g}{\det \mathbf{C}} \left(\frac{E_J E_L}{4E_{C,Q}E_{C,r}}\right)^{1/4} \\
&= -\frac{e^2 C_g}{\det \mathbf{C}} \left(\frac{2E_L}{E_{C,r}}\right)^{1/4} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4} \\
&= -\frac{e^2 C_g}{\det \mathbf{C}} \sqrt{\frac{1}{2E_{C,r}} \left(8E_L E_{C,r}\right)^{1/2}} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4} \\
&= -\frac{e^2 C_g}{\det \mathbf{C}} \sqrt{\frac{\hbar\omega_r}{2E_{C,r}}} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4} \\
&= -\frac{e^2 C_g}{\det \mathbf{C}} \sqrt{\frac{\widetilde{C}_r}{e^2}} \sqrt{\hbar\omega_r} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4} \\
&= -\frac{e C_g}{\det \mathbf{C}} \sqrt{\frac{\det \mathbf{C}}{C_Q + C_g}} \sqrt{\hbar\omega_r} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4} \\
&= \boxed{-\frac{C_g}{\sqrt{C_Q + C_g}} \sqrt{\frac{\hbar\omega_r e^2}{\det \mathbf{C}}} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4}.}
\end{aligned}$$

To see it gives us the result

$$g = -\frac{C_g}{C_Q} \sqrt{\frac{\hbar\omega_r e^2}{C_r}} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4}$$

when $C_g \ll C_Q, C_r$, we first see that

$$\begin{aligned}
\widetilde{C}_r &= \frac{\det \mathbf{C}}{C_Q + C_g} \\
&= \frac{C_Q C_r + C_Q C_g + C_r C_g}{C_Q + C_g} \\
&= \frac{C_r\left(C_Q + C_g\right) + C_Q C_g}{C_Q + C_g} \\
&= C_r + \frac{C_Q C_g + C_g^2 - C_g^2}{C_Q + C_g} \\
&= C_r + \frac{C_g(C_Q + C_g)}{C_Q + C_g} - \frac{C_g^2}{C_Q + C_g} \\
&\approx C_r + C_g \\
&\approx C_r.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\frac{C_g}{\det \mathbf{C}} &= \frac{C_g}{C_Q C_r + C_Q C_g + C_r C_g} \\
&= \frac{C_g}{\left(C_Q + C_g\right)\left(C_r + C_g\right) - C_g^2} \\
&\approx \frac{C_g}{\left(C_Q + C_g\right)\left(C_r + C_g\right)} \\
&\approx \frac{C_g}{C_Q C_r}.
\end{aligned}$$

Then,

$$g = -\frac{e^2 C_g}{\det \mathbf{C}} \sqrt{\frac{\widetilde{C}_r}{e^2}} \sqrt{\hbar\omega_r} \left(\frac{E_J}{8E_{C,Q}}\right)^{1/4}$$

$$\approx -\frac{C_g}{C_Q C_r}\sqrt{C_r}\sqrt{\hbar\omega_r e^2}\left(\frac{E_J}{8E_{C,Q}}\right)^{1/4}$$

$$= -\frac{C_g}{C_Q}\sqrt{\frac{\hbar\omega_r e^2}{C_r}}\left(\frac{E_J}{8E_{C,Q}}\right)^{1/4}.$$

If the approximation is taken much earlier in the derivation, the defination of $E_{C,Q}$ will also now include the approximated capacitance

$$E_{C,Q} = \frac{e^2}{2\widetilde{C}_Q} \approx \frac{e^2}{2(C_Q + C_g)} \approx \frac{e^2}{2C_Q}.$$